601. Multiplying out, using the binomial expansion:

$$x^3 + 6x^2 + 12x + 8 - (x^3 - 6x^2 + 12x - 8) = 0.$$

The odd-powered terms cancel, giving a quadratic $x^2 = -4/3$. Since the RHS is negative, this has no real roots.

602. The possibility space is:

	1	2	3	4	5	6
1						✓
2					✓	
3				✓		
4			✓			
5		✓				
6	✓					

In the possibility space, outcomes producing the same sum form diagonal lines. The longest of these is, as shown, the diagonal corresponding to a total of 7. So, this is the most likely sum.

603. Multiplying up by $\left(\sqrt{x}+1\right)$

$$\begin{aligned} &\frac{\sqrt{x}-1}{\sqrt{x}+1} = 2 - \sqrt{x} \\ \implies &\sqrt{x} - 1 = \left(2-\sqrt{x}\right)\left(\sqrt{x}+1\right) \\ \implies &\sqrt{x} - 1 = 2 + \sqrt{x} - x \\ \implies &-1 = 2 - x \\ \implies &x = 3 \end{aligned}$$

604. The parametric equations are $x = t$, $y = 1 + \frac{1}{2}t$. Substituting these into the equation of the circle,

$$\begin{aligned} &t^2 + \left(1 + \tfrac{1}{2}t\right)^2 = 1 \\ \implies &\tfrac{5}{4}t^2 + t = 0 \\ \implies &t\left(\tfrac{5}{4}t + 1\right) = 0 \\ \implies &t = 0, -\tfrac{4}{5}. \end{aligned}$$

So, the line is inside the circle for $t \in \left(0, -\frac{4}{5}\right)$. This interval has width $\frac{4}{5}$. The t-domain of the line segment is $[-2, 1]$, which has width 3. Hence, $\frac{4}{5} \div 3 = \frac{4}{15}$ of the length of the line lies within the circle, as required.

605. Using the index law $\left(a^b\right)^c \equiv \left(a^c\right)^b$,

$$16^t = \left(8^{\frac{4}{3}}\right)^t = \left(8^t\right)^{\frac{4}{3}} = y^{\frac{4}{3}}.$$

606. These are a pair of linear simultaneous equations in x^2 and y^2. We can eliminate y^2 by adding the equations. This gives $2x^2 = 3$. Substituting back for x^2, we get $y^2 = -\frac{1}{2}$. Since y^2 must be non-negative, this has no real roots.

607. NII gives simultaneous equations:

$$\begin{aligned} 3Q - 2P &= 16 \\ 5Q - 6P &= 8. \end{aligned}$$

Solving these gives $P = 7$, $Q = 10$ (Newtons).

— Nota Bene —

In this question, it is reasonable to answer with or without units of Newtons. It depends whether you think of $2P$ as representing the unitless number 14 or 14 Newtons. Often (but not in this question), a force will be presented as having magnitude F N. In that case, if asked for the value of F, it would be technically incorrect to add units of Newtons.

608. We require the inputs $1 - x^i$ of the square root function to be non-negative. So, we need to solve inequalities $1 - x^i \geq 0$, for $i = 1, 2, 3$. The largest real domains are

(a) $\{x : x \leq 1\}$,
(b) $\{x : -1 \leq x \leq 1\}$,
(c) $\{x : x \leq 1\}$.

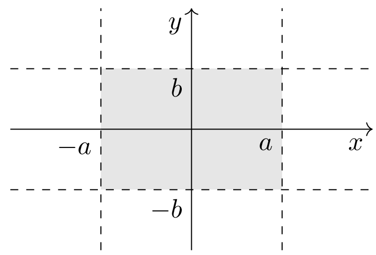
609. Starting with the LHS,

$$\begin{aligned} &\tan\theta + \cot\theta \\ \equiv\; &\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\ \equiv\; &\frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \\ \equiv\; &\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}. \end{aligned}$$

Using the first Pythagorean trig identity, this is

$$\begin{aligned} &\frac{1}{\sin\theta\cos\theta} \\ \equiv\; &\operatorname{cosec}\theta\sec\theta, \text{ as required.} \end{aligned}$$

610. The region described is a rectangle, bounded by the lines $x = \pm a$, $y = \pm b$.



Hence, the area is $4ab$.

— Nota Bene —

The area would be the same if the boundary lines were included. That is because a line, being one-dimensional, is infinitely thin and has no area.

611. The circle $x = \cos t$, $y = \sin t$ has been stretched by factor 4 in the x direction and by 3 in the y direction. Hence, the greatest diameter is 8 (in x), and the least is 6 (in y).

612. The formula for the sum of the first n integers is

$$S_n = \tfrac{1}{2}n(n+1).$$

Substituting $n = k$ and equating to $3k$,

$$\begin{aligned} &\tfrac{1}{2}k(k+1) = 3k \\ \implies &\tfrac{1}{2}k^2 - \tfrac{5}{2}k = 0 \\ \implies &k = 0, 5. \end{aligned}$$

$k = 0 \notin \mathbb{N}$, so $k = 5$.

613. The factor theorem concerns *linear* factors $(x-\alpha)$. So, to establish whether (x^2+1) is a factor, we would have to factorise it. The equation $x^2+1=0$ has no real roots, however. So, the factor theorem gives us no help.

— Nota Bene —

The factor theorem could be used over the complex numbers $\mathbb{C}$; the technique is beyond the scope of this book. It gives the name i to $\sqrt{-1}$. Then, over $\mathbb{C}$, the quadratic x^2+1 can be factorised as $(x+i)(x-i)$. So, the quintic has factors of $(x \mp i)$ iff it has roots of $x = \pm i$. Evaluating,

$$4x^5 + x + 1 \Big|_{x=\pm i} = \pm 5i + 1 \neq 0.$$

So, $4x^5 + x + 1$ does not have a factor of x^2+1.

614. The angle between the two directions of travel is 100°. So, using the cosine rule, the speed at which the two ships are separating is given by

$$v^2 = 14^2 + 16^2 - 2 \cdot 14 \cdot 16 \cos 100°.$$

Taking the positive root, $v = 23.01726...$ mph. For the ships to separate by 10 miles, it takes

$$\frac{10}{23.01726...} \approx 26 \text{ minutes}.$$

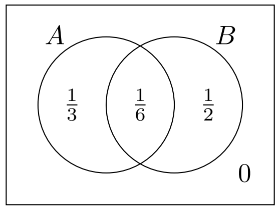
615. The boundary equation is $x^2 - x = 0$, which has solution set $\{0, 1\}$. Hence, the set given consists of all values between these, inclusive of the endpoints. In interval set notation, this is $[0, 1]$.

616. Consider the sequences of odd and even numbers, which are both APs. The products of these are 1×2, 3×4, 5×6. Since $12 - 2 \neq 30 - 12$, the sequence of products is not an AP. This disproves the claim.

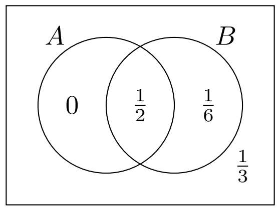
617. A fraction in its lowest terms is zero if and only if its numerator is zero, giving $x = \pm a$. So, if this equation is to have fewer than two real roots, then either the fraction must not be in its lowest terms, or the numerator must have fewer than two real roots. The first condition is $a = b$, and the second is $a = 0$.

618. Consider the possible values of $\mathbb{P}(A \cap B')$, i.e. the amount of the set A that lies outside the set B. This is minimised with $A \subset B$ and maximised with $\mathbb{P}(A' \cap B') = 0$. These boundary cases are

① $\mathbb{P}(A \cap B) = \frac{1}{6}$:



② $\mathbb{P}(A \cap B) = \frac{1}{2}$:



So, the possible values are

$$\mathbb{P}(A \cap B) \in \left[\tfrac{1}{6}, \tfrac{1}{2}\right].$$

619. By inspection, $x = 1$ is a root. So, by the factor theorem, $(x-1)$ is a factor. Taking this out,

$$\begin{aligned} &x^3 - 2x^2 - 5x + 6 \\ \equiv\ &(x-1)(x^2 - x - 6) \\ \equiv\ &(x-1)(x+2)(x-3). \end{aligned}$$

620. (a) The integral of $\mathrm{f}(x)$ is $\mathrm{F}(x)$ $(+c)$:

$$\begin{aligned} &\int_0^4 \mathrm{f}(x)\,dx \\ =\ &\Big[\mathrm{F}(x)\Big]_0^4 \\ =\ &\mathrm{F}(4) - \mathrm{F}(0) \\ =\ &4. \end{aligned}$$

(b) Integration is linear. So,

$$\begin{aligned} &\int_0^4 5\,\mathrm{f}(x) - 1\,dx \\ =\ &\Big[5\,\mathrm{F}(x) - x\Big]_0^4 \\ =\ &\big(5\,\mathrm{F}(4) - 4\big) - \big(5\,\mathrm{F}(0) - 0\big) \\ =\ &16. \end{aligned}$$

621. (a) The rectangular grid has area 12. From this, we subtract four right-angled triangles of total area 5. So, the shaded parallelogram has area 7 square units.

(b) The vertices on the left and right must form a diameter. The distance between them is $\sqrt{4^2+1^2}=\sqrt{17}$. So, the circumcircle has area

$$A=\pi\left(\tfrac{\sqrt{17}}{2}\right)^2=\tfrac{17}{4}\pi.$$

622. The statement does not hold. In the Newtonian model, for an object to remain in equilibrium, it is required that it has not only no resultant force on it, but also no resultant *moment*. Objects with no resultant force acting on them may still experience angular acceleration.

623. If α is a fixed point of the iteration, then α is a root of the equation

$$x=x^2-\frac{3}{x+1}.$$

Multiplying by $x+1$ gives

$$\begin{aligned}&x^2+x=x^3+x^2-3\\ \Longrightarrow\ &x^3-x-3=0.\end{aligned}$$

So, because $x=\alpha$ is a root of $x^3-x-3=0$, the factor theorem tells us that $(x-\alpha)$ must be a factor of x^3-x-3, as required.

624. We multiply top and bottom of the large fraction by the denominator of the small fractions. This gives an initial simplification

$$\begin{aligned}y&=\frac{1+\frac{1}{x+1}}{1-\frac{1}{x+1}}\\ &\equiv\frac{(x+1)+1}{(x+1)-1}\\ &\equiv\frac{x+2}{x}.\end{aligned}$$

We then make x the subject:

$$\begin{aligned}&xy=x+2\\ \Longrightarrow\ &x(y-1)=2\\ \Longrightarrow\ &x=\frac{2}{y-1},\text{ for } y\neq 1.\end{aligned}$$

625. Splitting the fraction up,

$$\begin{aligned}&\int_1^2 16x^{-2}+48x^{-3}\,dx\\ =\ &\Big[-16x^{-1}-24x^{-2}\Big]_1^2\\ =\ &\big(-8-6\big)-\big(-16-24\big)\\ =\ &26.\end{aligned}$$

626. (a) For f to be well defined, we require $1-x^2>0$, with strict inequality to avoid division by zero. So, the largest real domain is $(-1,1)$.

(b) The maximum of $1-x^2$ so 1, so the minimum of f is $\frac{1}{1}=1$.

(c) The denominator has range $(0,1]$, so $\mathrm{f}(x)$ has range $[1,\infty)$.

627. (a) The centre of the circle is equidistant from $(0,0)$ and $(4,0)$, which means it must lie on their perpendicular bisector $x=2$.

(b) Using Pythagoras, squared distance between $(4,0)$ and $(2,y)$ is

$$\begin{aligned}r^2&=(4-2)^2+(0-y)^2\\ &\equiv 4+y^2.\end{aligned}$$

Likewise between $(-2,6)$ and $(2,y)$,

$$\begin{aligned}r^2&=(-2-2)^2+(y-6)^2\\ &\equiv 16+(y-6)^2.\end{aligned}$$

(c) Equating the values of r^2,

$$\begin{aligned}&4+y^2=16+(6-y)^2\\ \Longrightarrow\ &4=16+36-12y\\ \Longrightarrow\ &y=4.\end{aligned}$$

The squared radius is therefore 20 and the area is 20π.

628. Taking out a common factor of $(x+1)$ directly, without multiplying out:

$$\begin{aligned}&(x+1)^3-4x^2-4x=0\\ \Longrightarrow\ &(x+1)\big((x+1)^2-4x\big)=0\\ \Longrightarrow\ &(x+1)(x^2-2x+1)=0\\ \Longrightarrow\ &(x+1)(x-1)^2=0\\ \Longrightarrow\ &x=\pm 1.\end{aligned}$$

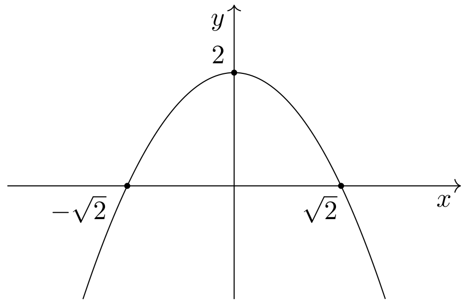
629. The second derivative of any quadratic function is constant. So, since $\mathrm{g}''(0)=-2$, we know that $\mathrm{g}''(x)=-2$ for all x. Integrating, $\mathrm{g}'(x)=-2x+c$. Substituting $\mathrm{g}'(0)=0$ gives $c=0$, so the first derivative is

$$\mathrm{g}'(x)=-2x.$$

Integrating again, $\mathrm{g}(x)=-x^2+d$. Substituting $\mathrm{g}(0)=2$ yields

$$\mathrm{g}(x)=-x^2+2.$$

The x intercepts are then $\pm\sqrt{2}$:



630. P' is the complement of P, i.e. not-P; $P \setminus Q$ is P minus Q, i.e. P with the elements of Q removed.

   (a) $(A' \cap B')' = A \cup B$.

   (b) $(A \cup B) \setminus (A' \cap B) = A$.

631. The right-hand set is $\left(-\frac{1}{2}, \frac{1}{2}\right)$, so the intersection is $\left(-\frac{1}{2}, 0\right]$.

632. Carrying out the definite integration,

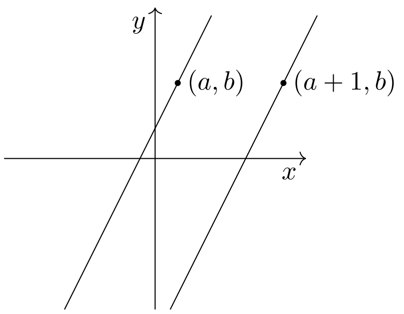
$$\begin{aligned} s &= \int_{T=0}^{T=t} u + aT\, dT \\ &= \left[uT + \tfrac{1}{2}aT^2\right]_{T=0}^{T=t} \\ &= ut + \tfrac{1}{2}at^2. \end{aligned}$$

— Nota Bene —

The reason for the unusual looking capital T here is that t is already in use as the total duration of the constant acceleration. T is then used to carry out the integration, meaning that it must take *all* values from $T = 0$ up to $T = t$. You could replace T with any variable (except s, u, a, t) and all would be well. A capital T is useful for understanding, however, as it keeps a connotation of time.

633. Multiplying by 2^x, we get $(2^x)^2 - 2 = 2^x$. This is a quadratic in 2^x. Rearranging and factorising, we have $(2^x - 2)(2^x + 1) = 0$. The latter factor has no roots, so $x = 1$.

634. These are parallel straight lines, with gradient m, through the points (a, b) and $(a + 1, b)$:



635. We know that $f'(x) = m$, for some constant m. So $f(x) = mx + c$. Substituting, this gives

$$\begin{aligned} & f(x + 1) - f(x - 1) \\ &= m(x + 1) + c - \big(m(x - 1) + c\big) \\ &\equiv mx + m + c - mx + m - c \\ &\equiv 2m. \end{aligned}$$

This is constant, as required.

636. The equation for intersections of the line $y = x$ and the parabola $y = g(x)$ is $x^2 - 3x + 3 = 0$. This has discriminant $\Delta = 9 - 12 < 0$. Hence, the positive parabola $y = g(x)$ is always above the line $y = x$. This means that the outputs of g (y values) are always bigger than the inputs of g (x values).

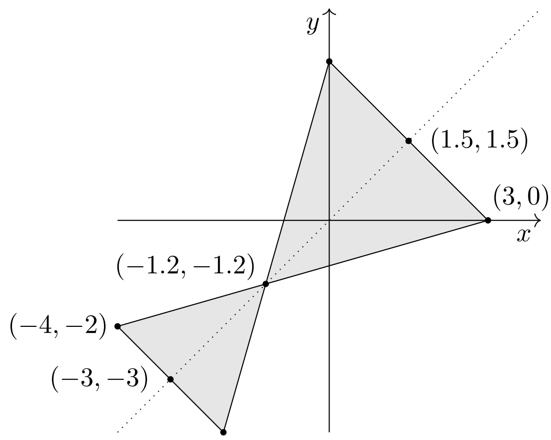
637. Setting up a quadratic equation and solving using the formula, we find that this expression is zero at

$$\begin{aligned} x &= \frac{842 \pm \sqrt{842^2 + 4 \cdot 680 \cdot 1207}}{2 \cdot 680} \\ &= \frac{71}{34}, -\frac{17}{20}. \end{aligned}$$

So, there are factors of $(34x - 71)$ and $(20x + 17)$. Since $34 \cdot 20 = 680$, there is no constant factor:

$$680x^2 - 842x - 1207 \equiv (34x - 71)(20x + 17).$$

638. The central intersection lies on the lines $y = x$ and $7y = 2x - 6$. Solving simultaneously gives $(-1.2, -1.2)$. Also finding the midpoints of the sides perpendicular to $y = x$, the scenario is



Each isosceles triangle is split into two right-angled triangles. Using $A_\triangle = \frac{1}{2}bh$, the total shaded area is given by

$$\begin{aligned} A_{\text{small}} + A_{\text{large}} &= \sqrt{2} \cdot 1.8\sqrt{2} + 1.5\sqrt{2} \cdot 2.7\sqrt{2} \\ &= 3.6 + 8.1 \\ &= 11.7 \text{ square units.} \end{aligned}$$

639. The exact value of $\tan^2 \frac{\pi}{6}$ is $\frac{1}{3}$. So, the percentage error is given by

$$\frac{\frac{\pi}{6}^2 + \frac{2}{3} \cdot \frac{\pi}{6}^4 - \frac{1}{3}}{\frac{1}{3}} \approx -2.7\%.$$

640. By Pythagoras, the diagonals of the rectangle have length $\sqrt{20^2 + 28^2} \approx 34.41$ cm. The diagonals of the rectangle will therefore not fit inside a circle of diameter 34 cm.

641. We know that $x = ay + b$ and $y = cz + d$, for some constants a, b, c, d. Substituting for y gives

$$\begin{aligned} & x = a(cz + d) + b \\ \Longleftrightarrow\ & x = (ac)z + (ad + b). \end{aligned}$$

This is a linear relationship, as required.

642. Starting with the RHS, we can write xyz as $x(yz)$ and use the assumed log rule:

$$\log_a \big(x(yz)\big) = \log_a x + \log_a (yz).$$

Using the same rule on the second term, this is

$$\log_a x + (\log_a y + \log_a z).$$

This proves the required result.

643. Start with $(x + 1)^2 \equiv x^2 + 2x + 1$. The x^2 term matches, but we have $2x$, where x is required. So, subtract $(x + 1)$. This gives a constant term of 0, when we want 6. Adding 6,

$$x^2 + x + 6 \equiv (x + 1)^2 - (x + 1) + 6.$$

——— Alternative Method ———

Let $y = x + 1$, so that $x = y - 1$. Substituting,

$$\begin{aligned} & x^2 + x + 6 \\ = {} & (y - 1)^2 + (y - 1) + 6 \\ \equiv {} & y^2 - y + 6 \\ = {} & (x + 1)^2 - (x + 1) + 6. \end{aligned}$$

644. Since $x = -a$ is clearly a root, the right-hand bracket must either have no real roots, or it must be identically equal to $(x + a)^2$.

- For the right-hand bracket to have no real roots, the discriminant must be negative. So, $16 - 4b < 0$, which gives $b \in (4, \infty)$.
- For the right-hand bracket to be identically equal to $(x+2)^2$, $2a$ must equal 4. This gives $a = 2$ and therefore $b = 4$.

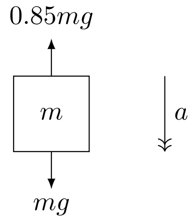
Combining the cases, the set of possible values of b is $(4, \infty) \cup \{4\} = [4, \infty)$.

645. Both are GPs. So, we use $S_\infty = \frac{a}{1-r}$. The first term is $a = 1$ in both cases, and the common ratio is $r = \pm\frac{1}{2}$. Since $|r| < 1$, both sums converge:

(a) $S_\infty = \dfrac{1}{1 - \frac{1}{2}} = 2$,

(b) $S_\infty = \dfrac{1}{1 + \frac{1}{2}} = \frac{2}{3}$.

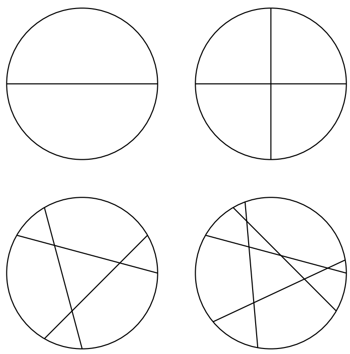
646. Weighing scales measure the reaction force applied downwards on them. This force is the NIII pair of the reaction force applied upwards on the object being weighed. In the accelerating lift, the scales measure an object of weight mg N as having weight $0.85mg$ N. So, the reaction force on an object of mass m is $0.85mg$ N. The force diagram is



NII is $mg - 0.85mg = ma$. The masses cancel. The acceleration is $0.15g$ ms^{-2} downwards.

647. Since division by $(x-3)$ leaves no remainder, $(x-3)$ is a factor of the quadratic. Hence, by the factor theorem, $x = 3$ is a root. So, $9p + 3 + p = 0$, which gives $p = -\frac{3}{10}$.

648. The maximal number of pieces occurs if every pair of cuts intersect, and no three cuts are concurrent. In the cases $n = 1, 2, 3, 4$, this gives $P_n = 2, 4, 7, 11$:



The formula gives the same values.

——— Alternative Method ———

For $n = 1, 2, 3, 4$, the formula gives $2, 4, 7, 11$. To verify this, we note that the case $P_0 = 1$ is trivial. After that, the $(n+1)$th cut can cross a maximum of n lines, which gives creation of a maximum of $(n + 1)$ new pieces. So,

$$\begin{aligned} P_0 &= 1 \\ P_1 &= 1 + 1 = 2, \\ P_2 &= 2 + 2 = 4 \\ P_3 &= 4 + 3 = 7, \\ P_4 &= 7 + 4 = 11. \end{aligned}$$

This verifies the ordinal formula.

649. L has equation $2x + 3y = k$, and passes through point $(-6, 2)$. Hence, its equation is $2x+3y = -6$. Substituting the point $(6, -6)$ into the LHS gives $2 \cdot 6 + 3 \cdot -6 = -6$, so L goes through $(6, -6)$.

650. Cubics give simple counterexamples: every cubic has one point of inflection, which is at its centre of rotation. This may be stationary or not.

    (a) The origin in $y = x^3 + x$ is a counterexample. The first derivative is $3x^2 + 1$, with a gradient of 1; the second derivative is $6x$, which changes sign at the origin.

    (b) The origin in $y = x^3$ is a counterexample. The first derivative is $3x^2$, giving a gradient of 0 at the origin; the second derivative is $6x$, which changes sign at the origin.

651. The loci are straight lines. Their gradients are a and $-\frac{1}{a}$, which are negative reciprocals, so they intersect normally. Furthermore, at $x = b$, each equation gives $y = c$. So, they intersect normally at (b, c), as required.

652. The answer here is yes and no, depending on your definitions. Forces are mathematical abstractions, not tangible objects, so the definitions are relevant.

    (a) If forces *internal* to an object are considered, then both parts of a Newton pair may be seen to act on a single object. For example, if I press my hands together, then the two parts of the Newton pair may both be seen as acting on me. But, in this case, the two forces must cancel out by definition. And, since they have no resultant effect, it would be equally correct to say that there are no such forces.

    (b) If only *external* forces, i.e. forces which have a resultant effect, are considered, then the two forces must act on different objects. This is the standard use of a Newton pair, in which A acts on B and B acts on A.

653. (a) At $t = 0$, $\frac{d}{dt}(k) = 0.32\text{s}^{-1}$.

    (b) The pH begins to fall from the moment at which the rate of change drops to zero. Solving $0.32 - 0.012t = 0$ gives $t = 26.7$ s (3sf).

    (c) Δk is the total change in pH value. This is given by the definite integral of the rate of change of k:

$$\begin{aligned}\Delta k &= \int_0^{60} 0.32 - 0.012t \, dt \\ &= \Big[0.32t - 0.006t^2\Big]_0^{60} \\ &= 0.32 \cdot 60 - 0.006 \cdot 60^2 \\ &= -2.4\end{aligned}$$

    (d) The rate of change operator $\frac{d}{dt}$ is a division (in the infinitesimal limit) by time t. So, it has units of s^{-1} i.e. "per second". Since $\frac{d}{dt}(k)$ also has units s^{-1}, k cannot have units.

654. If $\text{F}'(x) = \text{f}(x)$, then

$$\int_a^b \text{f}(x)\, dx = \text{F}(b) - \text{F}(a).$$

Reversing the limits gives $\text{F}(a) - \text{F}(b)$, which is equal to $-\big(\text{F}(b) - \text{F}(a)\big)$. Hence, if the limits of a definite integral are switched, then the value of the integral is negated. □

655. A hexagon consists of six equilateral triangles. So, there are 12 shaded triangles and, in the hexagons, $6 \times 6 = 36$ unshaded triangles. Hence, the fraction of the area covered by the triangles is $\frac{12}{36} = \frac{1}{3}$.

656. Quoting the standard formulae, the mean is

$$\bar{x} = \frac{\sum x}{n} = \frac{327}{25} = 13.08,$$

and the standard deviation is

$$s_x = \sqrt{\frac{5229 - 25 \cdot 13.08^2}{25}} = 6.17 \text{ (3sf)}.$$

657. Rearranging $x^2 - 2y^2 = 1$ to make 2 the subject,

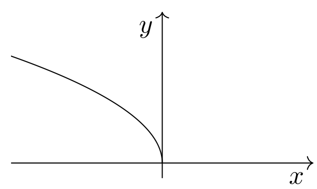
$$2 = \frac{x^2 - 1}{y^2}.$$

Since y^2 is positive, so is $x^2 - 1$. And y is also positive. Hence, we can take the positive square root of both sides, giving

$$\sqrt{2} = \frac{\sqrt{x^2 - 1}}{y}.$$

For a large positive integer x, the numerator $\sqrt{x^2 - 1}$ is approximately equal to x. This gives $\sqrt{2} \approx x/y$, as required.

658. The graph $y = \sqrt{-x}$ is a reflection of $y = \sqrt{x}$ in the y axis:



659. Upper-case Δt (Delta t) is a finite change in t, such as a period of 3 days. Lower-case δt (delta t) is also a finite change in t, but has the connotation of being small, such as a period of a microsecond. Latin dt is an infinitesimal change in t, and cannot be defined in isolation. It notates the limit of δt as $\delta t \to 0$, and only makes sense in either

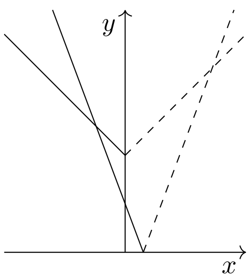
- comparison by ratio to other infinitesimal quantities, such as in $\frac{dx}{dt}$, or
- when summed infinitely in $\int \mathrm{f}(t)\, dt$.

660. Since the rationals $\mathbb{Q}$ are a subset of the real $\mathbb{R}$, the implication is $x \in \mathbb{Q}' \Longleftarrow x \in \mathbb{R}'$. If a number is non-real, then it cannot be rational.

661. Since two 1 cm radii fit into the shorter side of the rectangle, the region within 1 cm of a vertex is four quarter circles.

The rectangle has area 8 cm^2. The four quarter circles form one whole circle of radius 1 cm and area π cm^2. The fraction of the rectangle which is shaded is therefore $\frac{\pi}{8}$.

662. The graphs of $y = $ LHS and $y = $ RHS are as follows. Where the mod signs are passive, the graphs are dashed:



The intersections occur when the mod signs are (active, active) and (passive, passive). Hence, the equations are $-(3x-1) = 2-x$ giving $x = -1/2$, and $3x - 1 = 2 + x$ giving $x = 3/2$.

663. Factorising, we have $x = (y-4)^2$. The presence of squared factor on the RHS shows that the curve is tangent to the y axis at $y = 4$.

664. Using $\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y) - \mathbb{P}(X \cap Y)$,

$$\begin{aligned}\mathbb{P}(X \cap Y) &= \frac{1}{m} + \frac{1}{n} - \frac{m+n-1}{mn} \\ &\equiv \frac{n}{mn} + \frac{m}{nm} - \frac{m+n-1}{mn} \\ &\equiv \frac{1}{mn} \\ &= \mathbb{P}(X) \times \mathbb{P}(Y).\end{aligned}$$

We have shown that $\mathbb{P}(X \cap Y) = \mathbb{P}(X) \times \mathbb{P}(Y)$, which is equivalent to the statement that events X and Y are independent.

665. (a) This is well defined. Adding two to the outputs of g does not affect its domain of definition.
(b) This is not well defined. Adding two to the x values means that the original domain of $[0,1]$ would give inputs to g in the set $[2,3]$. The function g is not defined for these values.

666. Integrating once gives

$$v = \int a\, dt = at + c_1.$$

The constant c_1 is the value of v at $t = 0$, which is the initial velocity u. This gives $v = u + at$. Integrating again,

$$s = \int u + at\, dt = ut + \tfrac{1}{2}at^2 + c_2.$$

Since, in *suvat*, the displacement s is measured from the initial position, the constant c_2 is zero by definition. This gives $s = ut + \frac{1}{2}at^2$. QED.

667. The probability that the score on a die is prime is usually $3/6$. Knowing that the score is even changes the numerator to 1 and the denominator to 3. This knowledge decreases the relevant probability from $1/2$ to $1/3$.

668. (a) The equation of a general line, gradient m, through $(0,-4)$, is $y = mx - 4$.
(b) The equation for intersections is $x^2 = mx - 4$, which we can rearrange to $x^2 - mx + 4 = 0$. This is a quadratic in x. For tangency, it must have a double root. So, its discriminant must be zero.
(c) We require $\Delta = 0$, so $m^2 - 16 = 0$, giving $m = \pm 4$. Solving for x and substituting gives P as $(\pm 2, 4)$. To match the diagram, P must be $(2,4)$.

669. Given that $0! = 1$ by definition, this is

$$\left[\frac{x!}{1+2^{-x}}\right]_0^1 = \frac{1}{1+\frac{1}{2}} - \frac{1}{1+1} = \frac{1}{6}.$$

670. Since the numerator is fixed, the solution can only be different if the denominator shares factors with the numerator. This is the case if $k = \pm 2$.

671. The quartiles are defined by

$$\mathbb{P}(X < X_{\text{lower}}) = 0.25, \quad \mathbb{P}(X < X_{\text{upper}}) = 0.75.$$

Using the statistical facility on a calculator, the quartiles are given by $X_{\text{lower}} = -0.674489$ and $X_{\text{upper}} = 0.674489$. The IQR is then the difference, which is $2 \times 0.674489 = 1.35$ (3sf).

672. The quartic graph shown has range $[k, \infty)$, for some $k < 0$, while $y = (x-1)^4$ has range $[0, \infty)$.

— Alternative Method —

The quartic shown has its turning point at $x = 0$. But the equation $y = (x-1)^4$ has its turning point at $x = 1$.

673. (a) Evaluating the definite integral,

$$\int_1^N \frac{1}{x^2}\,dx \equiv \left[-x^{-1}\right]_1^N \equiv 1 - \frac{1}{N}.$$

(b) The infinite integral is defined as

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x^2}.$$

From part (a), this is $\lim_{N\to\infty} 1 - \frac{1}{N} = 1$.

— Nota Bene —

This question tells us that it is possible to have a two-dimensional region of the (x, y) plane that is infinite in extent but finite in area.

674. This is a GP, so the ratio between successive terms is constant. Expressed algebraically, this is

$$\begin{aligned} \frac{a+2}{a} &= \frac{a+3}{a+2} \\ \Longrightarrow (a+2)^2 &= a(a+3) \\ \Longrightarrow a^2+4a+4 &= a^2+3a \\ \Longrightarrow a &= -4. \end{aligned}$$

675. This is true by definition, irrespective of whether the car is turning a corner or not.

① Reaction forces are defined as contact forces *perpendicular* to the relevant surfaces.

② Frictional forces are defined as contact forces *parallel* to the relevant surfaces.

Reaction and friction are always perpendicular.

— Nota Bene —

This modern usage of the word "reaction" does not tie in exactly with Newton's original usage, in which he juxtaposed *actions* and *reactions*. The words have changed meanings in the interim. In modern parlance, a Newton (NIII) pair can consist of either zero reaction forces or two reaction forces, but can never consist of one.

676. (a) $\frac{dy}{dx} = kx^2$,

(b) $6 = 3k$, so $k = 2$,

(c) Using the results of parts (a) and (b),

$$\int 2k^2\,dx = \tfrac{2}{3}x^3 + c.$$

(d) The information $x = 0$, $y = 6$ gives $c = 6$. So, when $x = 3$, $y = \frac{2}{3}\cdot 3^3 + 6 = 24$.

677. (a) A projectile is modelled as a particle (object of negligible size) which has acceleration g ms^{-2} vertically downwards.

— Nota Bene —

This is equivalent to tated as saying that a projectile is a particle on which the only force acting is gravity.

(b) $14\cos 30° = 7\sqrt{3}$ and $14\sin 30° = 7$ ms^{-1}.

(c) Vertically, $0 = 7t - 4.9t^2$, which has roots $t = 0$ and $t = \frac{10}{7}$. The former is take-off, the latter is landing.

(d) Since horizontal acceleration is zero, the range is $7\sqrt{3}\cdot\frac{10}{7} = 10\sqrt{3}$ m.

678. The discriminant is $\Delta = b^2 - 4ac$. Negation of the x term means negation of the value of b, which doesn't affect the value of b^2. Hence, these two equations have the same Δ and thus the same number of roots.

— Alternative Method —

To transform from the parabola $y = px^2 + qx + r$ to the parabola $y = px^2 - qx + r$, we replace x by $-x$. This is reflection in the y axis, which does not affect the number of intersections with the x axis.

679. The equation for intersections is $x^3 - 3x - 2 = 0$. By inspection, this is satisfied by $x = -1$. Hence $(x+1)$ must be a factor. Taking it out,

$$\begin{aligned} (x+1)(x^2-x-2) &= 0 \\ \Longrightarrow (x+1)^2(x-2) &= 0. \end{aligned}$$

Since this equation has a double root at $x = -1$, the line must be tangent to the cubic at $x = -1$.

680. The set of points equidistant from a pair of lines consists of their angle bisectors. In this case, this is the x and y axes, i.e. $y = 0$ and $x = 0$. A point is on the locus if either $y = 0$ or $x = 0$. This may, according to the factor theorem, be written as a single equation: $xy = 0$.

681. The LHS is the sum of the probabilities of the union and the intersection of A and B. This consists of elements in exactly one of A or B counted once, and elements in both A and B counted twice. This same single and double counting can be expressed as $\mathbb{P}(A) + \mathbb{P}(B)$. □

— Nota Bene —

The formula in this question is a rewriting of the standard inclusion-exclusion formula

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

682. Setting $x = \cos y$,

$$\begin{aligned} &\sin^2 y + \cos^2 y \equiv 1 \\ \implies &\sin y \equiv \pm\sqrt{1 - \cos^2 y} \\ \therefore \quad &\sin y = \pm\sqrt{1 - x^2}. \end{aligned}$$

The range of $\arccos x$ is $[0, \pi]$, so the range of $\sin(\arccos x)$ is $[0, 1]$. We can therefore take the positive square root. This gives

$$\sin(\arccos x) \equiv \sqrt{1 - x^2}.$$

683. (a) The other formula for the sum of squares is

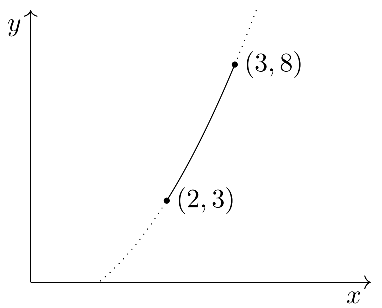
$$S_{xx} = \sum x^2 - \tfrac{1}{n}\left(\sum x\right)^2.$$

Taking values from the calculator,

$$S_{xx} = 1308 - \tfrac{1}{91} \times 310^2 = 252 \text{ (3sf)}.$$

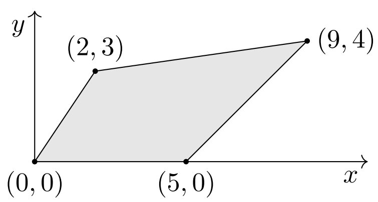
(b) The variance can be read off directly, as 2.77 (3sf), or calculated as $S_{xx}/91$.

684. The domain does not include the vertex of the quadratic. So, the function is increasing on $[2, 3]$.



Hence, the range is $[\mathrm{f}(2), \mathrm{f}(3)]$, which is $[3, 8]$.

685. When rotated, the quadrilateral is

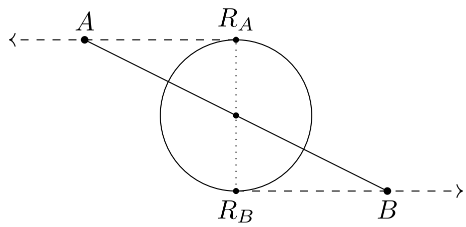


The vertices of a quadrilateral should be named in order around the perimeter. The student has not done this. Correct notation would be e.g. $ABCD$, where A is $(0, 0)$, B is $(5, 0)$, C is $(9, 4)$ and D is $(2, 3)$.

686. We reduce the fraction to its lowest terms before taking the limit:

$$\begin{aligned} &\lim_{k \to 2} \frac{4k^2 - 16}{k^2 - 2k} \\ = &\lim_{k \to 2} \frac{4(k+2)(k-2)}{k(k-2)} \\ = &\lim_{k \to 2} \frac{4(k+2)}{k} \\ = &\ 8. \end{aligned}$$

687. Newton III tells us that one astronaut letting go is equivalent to both astronauts letting go. So, the astronauts will move off along opposite tangents to the circle of rotation, at points of release R_A and R_B:



After release, each moves with speed v, covering a distance vt in t seconds. So, after t seconds they will have separated, in the tangential direction, by $2vt$. They remain a distance d apart in the radial direction. By Pythagoras, therefore, they will be a distance $\sqrt{4v^2t^2 + d^2}$ apart, as required.

688. If $\mathrm{f}'(x) = a$, then $\mathrm{f}(x) = ax + b$. This is a general formula for any linear function.

689. Rearranging and using the quadratic formula:

$$\begin{aligned} &4\sin^2 18° + 2\sin 18° - 1 = 0 \\ \implies &\sin 18° = \frac{-2 \pm \sqrt{4 - 4 \cdot 4 \cdot -1}}{2 \cdot 4} \\ \implies &\sin 18° = \frac{-1 \pm \sqrt{5}}{4}. \end{aligned}$$

Since $18°$ is an acute angle, we need the positive root, so $\sin 18° = \frac{-1+\sqrt{5}}{4}$.

690. Adding 2 to the t-expression for x and adding 3 to the t-expression for y gives the new equation as $x = 3 - \lambda$, $y = 6 + 2\lambda$, for $\lambda \in \mathbb{R}$.

— Alternative Method —

Replacing x by $x - 2$ and replacing y by $y - 3$ gives $x - 2 = 1 - \lambda$, $y - 3 = 3 + 2\lambda$, for $\lambda \in \mathbb{R}$. Making x and y the subjects, we get $x = 3 - \lambda$, $y = 6 + 2\lambda$, for $\lambda \in \mathbb{R}$.

691. Three values in AP must be symmetrical around the central value. Since the sum of the interior angles of a triangle is π radians, the middle angle must have value $\frac{\pi}{3}$ radians.

692. The derivative is

$$\frac{dy}{dx} = 2x - 10.$$

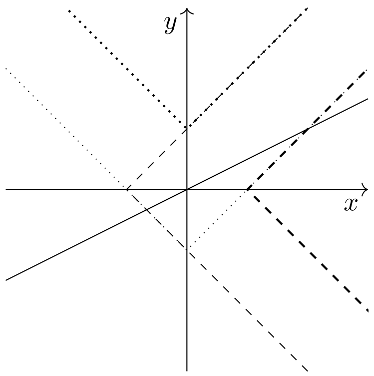
This must be non-negative, so we require

$$\begin{aligned} & 2x - 10 \geq 0 \\ \Longrightarrow\ & x \in [5, \infty). \end{aligned}$$

693. The function is ill defined when $x^2 + px + q \leq 0$. We are given that this is for $x \in [-3, 4]$. Hence, the quadratic must be $(x+3)(x-4) \equiv x^2 - x - 12$. So, $p = -1$ and $q = -12$.

694. If $x + y$ is constant, then $x + y = k$ for some $k \in \mathbb{R}$. Differentiating, we get $1 + \frac{dy}{dx} = 0$. So, $\frac{dy}{dx} = -1$.

695. The graphs are



The line $x - 2y = 0$ intersects each of the mod graphs except $y = |x| + 1$.

(a) No,

(b) Yes,

(c) Yes,

(d) Yes.

696. We know that $p - q = \frac{6}{10}(p+q)$, which tells us that $p = 4q$. The difference $p - q$ is then $3q$, while the geometric mean $\sqrt{pq}$ is $2q$. So, $p - q = \frac{3}{2}\sqrt{pq}$. The difference is three halves of the geometric mean, as required.

697. The transformation is a translation by vector $3\mathbf{i}$.

——— Nota Bene ———

Describing such a transformation as a “shift”, while fully comprehensible, is informal language. So, for instance, the LaTeX in which this book and its diagrams are typeset uses the code `xshift=1cm` to enact a translation. The formal mathematical term, however, is “translation”.

698. (a) NII gives $T - 13 = 3a$ and $65 - T = 10a$. Adding these gives $52 = 13a$, so $a = 4$ and $T = 25$.

(b) Since the accelerations of the two objects are different, the string connecting them must be capable of extension.

699. This is an infinite GP, with first term $a = 1024$ and common ratio $r = \frac{1}{2}$. Hence, the sum is

$$S_\infty = \frac{1024}{1 - \frac{1}{2}} = 2048.$$

700. (a) Completing the square gives

$$(x+2)^2 + (y-3)^2 = 13.$$

So, the radius is $\sqrt{13}$.

(b) The diagonals of the square are diameters of the circle. They have length $2\sqrt{13}$. Scaling this down by $1/\sqrt{2}$, the square has side lengths $l = 1/\sqrt{2} \times 2\sqrt{13}$. Squaring this, the area is

$$l^2 = \left(\frac{2\sqrt{13}}{\sqrt{2}}\right)^2 = \frac{4 \cdot 13}{2} = 26.$$

——— End of 7th Hundred ———